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Mathematics. — "*Central Projection in the space of* LOBATSCHEFSKY". (1st part). By Prof. H. DE VRIES. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of September 30, 1905).

1. Let an arbitrary plane τ be given in hyperbolic space; let the perpendicular be erected in an arbitrary point O_1 of τ, and let finally an arbitrary point O be taken on this perpendicular. We can now ask what we can notice if we project the figures of space out of O as a centre of projection on τ as a plane of projection or picture plane, and inversely, how the exact position and situation of the figures in space can be determined by means of their projections.

In the following a few observations will be given on these two questions.

Let us suppose an arbitrary plane ε through the line OO_1, standing therefore at right angles to τ; in that plane we can draw through O two straight lines p_1, p_2 parallel to the line of intersection e of ε and τ passing through O_1, therefore also parallel to τ itself. The angles formed by p_1 and p_2 with OO_1 are equal; they are both acute, and their amount is a function of the distance $OO_1 = d$. LOBATSCHEFSKY has called each of these two angles the *parallel angle* [1]) belonging to the distance d, and has indicated them by $\Pi_{(d)}$; if d is given, the parallel angle is found out of the relation

$$tg\ {}^1/_2\ \Pi_{(d)} = e^{-d},$$

in which for the number e the basis of the natural logarithmic system may be taken, if only the unity of length by which OO_1 is measured be taken accordingly [2]). As far as the range of values of $\Pi_{(d)}$ is concerned, I only observe that the parallel angle $= {}^1/_2\,\pi$ for $d = 0$, decreasing and tending to 0 if d increases and tends to ∞.

If the plane ε rotates round OO_1, then p_1 and p_2 will describe a cone of revolution round OO_1 as axis; this cone is the locus of all straight lines through O parallel to τ, and distinguishes itself in many respects, in form and properties, from the cone of revolution of Euclidian Geometry; the plane τ is an asymptotic plane, to which its surface tends unlimited, and from the symmetry with respect to O follows easily that another plane τ^* like this exists, also placed perpendicularly on OO_1, but in the point O_1^* situated symmetrically to O_1 with respect to O. So the cone is entirely included between the two planes τ and τ^*, and these two planes having not a single point in common (neither at finite nor at infinite distance), are *divergent;* however, they possess the common perpendicular $O_1O_1^*$, and their shortest distance is $2d$. The cone discussed here will be called for convenience, sake the parallel cone $\varkappa$ belonging to the point O.

2. The parallel cone divides the space into three separate parts; let us call those two parts, inside which is the axis OO_1, the interior of the cone, the remaining part the exterior; it is then easy to see that the points of space behave differently with respect to their projectability according to their lying inside or outside the cone; for a point P inside the cone the projecting ray OP forms with

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[1]) F. ENGEL: „N. I. LOBATSCHEFSKY. Zwei geometrische Abhandlungen". Leipzig, TEUBNER, 1899, p. 167.

[2]) F. ENGEL, l. c. p. 214.

OO_1 [1]) an acute angle smaller than the parallel angle, from which ensues that OP (or perhaps the prolongation of PO over O) must meet the picture plane; the point of intersection P' is the central projection of P. However, for P outside the cone the acute angle between OP and OO_1 is greater than the parallel angle; so now OP is divergent with respect to τ, from which ensues that points outside the cone possess no projections at all; points on the cone on the contrary do, but these projections lie at infinity.

From the fact that points outside the parallel cone are not projectible we need not infer that these points cannot be determined by Central Projection; if such a point is regarded let us say as the point of intersection of two straight lines, and if these *can* be projected by Central Projection, their point of intersection will also be determined in this indirect way.

3. Let a straight line l be at right angles to τ in a point D of τ. As the line OO_1 is also perpendicular to τ, it is possible to bring a plane through l and OO_1, the trace of which connects O_1 with D. This plane will intersect the cone $\varkappa$ in two generatrices, p_1, p_2; we now assume that l cuts these two lines in two points P_1 and P_2, and — to fix our thoughts — that P_1 lies between D and P_2. The line l possesses two points at infinity, $V_{1\infty}$, $V_{2\infty}$, which both lie inside the parallel cone; let us suppose that $V_{1\infty}$ lies under the picture plane and $V_{2\infty}$ above it, then the succession of the points on l is this: $V_{1\infty}$, D, P_1, P_2, $V_{2\infty}$.

The projecting ray $OV_{1\infty}$ cuts τ in a point V'_1 of e lying between D and O_1; we shall call it the first *vanishing point* of l. In like manner the ray $OV_{2\infty}$ prolonged over O will cut the line e in a point V'_2 lying in such a way that O_1 lies between V'_1 and V'_2; we shall call V'_2 the second vanishing point of l. The point O_1 does not lie in the middle between V'_1 and V'_2, on the contrary it is closer to V'_1; if namely we let down the perpendicular OS out of O to l a quadrangle is formed with three right angles, namely at O_1, D and S, and from this ensues that the fourth $\angle SOO_1$ is acute. Now OS is the bisectrix of $\angle V_{1\infty}OV_{2\infty}$, and therefore the perpendicular in O on OS the bisectrix of $\angle V'_1OV'_2$; this perpendicular must be placed, as $\angle SOO_1$ is acute, between OO_1, and OV'_2, and from this ensues $\angle V'_1OO_1 < \angle V'_2OO_1$. If we now let the rectangular triangle V'_1OO_1 rotate about the side OO_1 till it lies on the triangle V'_2OO_1, then we can immediately find that $O_1V'_1 < O_1V'_2$.

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[1]) By OO_1 we understand the straight line prolonged at both ends unlimitedly, by OP however a semi-ray starting from O.

It is clear that the central projection l' of l coincides with the trace e of the projecting plane $Ol = \varepsilon$, and at the same time that l is determined by its point of intersection and by *one* of the two vanishing points; the second will be found by letting down the perpendicular OS out of O to l, and by setting off at the other side of OS an angle equal to the parallel angle, formed by OS and the only existing ray parallel to l. But further can be remarked that l is also determined by its two vanishing points, or what comes to the same by its two points at infinity; to find l we should but have to bisect the angle formed by the two projecting parallel rays of l, to mark on the bisecting line a segment OS corresponding to $\frac{1}{2} \angle V_{1\infty} O V_{2\infty}$ as parallel angle, and to erect the perpendicular in S on OS.

The line l is divided into four segments by its two points at infinity, its point of intersection and its two points P_1, P_2 (whose projections lie at infinity), and l' in like manner by its points at infinity $P'_{1\infty}$, $P'_{2\infty}$, the point D, and the two vanishing points V'_1, V'_2 of l; the connection between these different segments of l and l' is as follows. To the infinite segment $V_{1\infty} D$ corresponds the finite segment $V'_1 D$, and to the finite segment DP_1 the infinite segment $DP'_{1\infty}$; to the points between P_1 and P_2 no projections correspond, because the projecting rays of these points are divergent with respect to τ; to the infinite segment $P_2 V_{2\infty}$ on the contrary a segment of l' again corresponds, namely the infinite segment $P'_{2\infty} V'_2$. There now remain on l' only the points between the two vanishing points, to which also belongs O_1; to these no points of l correspond, their projecting rays being divergent with respect to l.

§ 4. If a line $l \perp \tau$ is to cut the surface of the parallel cone in two points, the length of DO_1 may not exceed a certain upper limit, so that the results just found do not hold for *all* lines $\perp \tau$. Let us again suppose through OO_1 an arbitrary plane ε, and let us now first regard OO_1 itself. If we let down out of O_1 on to p_2 the perpendicular $O_1 T$, then because p_2 is parallel to e, the angle $TO_1 P'_{2\infty}$ is the parallel angle belonging to $O_1 T$, and therefore angle $TO_1 O$ is smaller than this parallel angle, because $O_1 O$ cuts the line p_2 (namely in $\angle O$); and $OO_1 P'_2$ being equal to 90°, the parallel angle $TO_1 P'_{2\infty} > 45°$, and angle $TO_1 O < 45°$. If in ε we move l, first coinciding with OO_1, in such a way that it remains in D perpendicular to e, namely towards the side of $P'_{1\infty}$ (therefore from $P'_{2\infty}$), then the perpendicular DT on p_2 becomes continually greater, and so (see N°. 1) the parallel angle $TDP'_{2\infty}$ continually smaller; as soon as the perpendicular DT has attained such a length that the

parallel angle corresponding to it is precisely 45°, the complement becomes 45° too, and therefore l parallel to p_2, but on the other side of DT compared to e; l will still intersect p_1 in a finite point P_1, for as it enters the triangle $OO_1P'_{1\infty}$ at D, does of course not contain the point $P'_{1\infty}$, and is divergent with reference to OO_1, it can leave the triangle only in a finite point of p_1; but it will cut p_2 in an point at infinity $P_{2\infty}$, being at the same time $V_{2\infty}$. So its projectio consists of the segment of the line e of V'_1 over D to $P'_{1\infty}$, and the isolated point $P'_{2\infty}$ is equal to $V'_{2\infty}$; now too it is determined by two of the three points D, V'_1, $V'_{2\infty}$.

The point D lies at a certain distance r from O_1; if we describe a circle in τ about O_1 as centre and with r as radius, and if we erect in all points of that circle the perpendiculars on τ, a surface appears which may be called a cylinder of revolution, of which the circle just mentioned is the gorge line; the lines l ($\perp\tau$) lying inside that cylinder have two different vanishing points (with the exception of OO_1, whose projection is a single point), the lines l on the cylinder have a finite and an infinite vanishing point, and the lines l outside the cylinder miss the second vanishing point.

As for the shape of the cylinder it is easy to see, that the plane τ^* (see N°. 1) is an asymptotic plane; and τ itself being evidently a plane of orthogonal symmetry, the plane τ^{**} normal to OO_1 in the point O_1^{**} symmetrical to O_1^* with respect to τ will be a second asymptotic plane; so the distance of these two planes is $4d$.

5. In Euclidean Geometry the lines $\perp\tau$ are at the same time those which are parallel to OO_1, but in Hyperbolic Geometry this is different; here we have to regard the lines having in common with OO_1 the point $V_{1\infty}$ lying under the picture plane at infinity, and those having with OO_1 in common the point $V_{2\infty}$ lying above τ. A line l of the former kind lying in the vicinity of OO_1 has a picture point D, two points P_1, P_2, and a second point at infinity lying inside the cone $\varkappa$; its first vanishing point coincides with O_1, whilst the second lies on DO_1 in such a way, that O_1 lies between D and that point.

If the perpendicular OS let down out of O to l becomes continually larger, the first particularity appearing here is that l becomes parallel to the generatrix p_2 of cone $\varkappa$ lying in the plane Ol; then it is at right angles to the bisectrix of the obtuse angle formed by p_2 and OO_1. All lines having this property form an asymptotic cone of revolution [1]) with vertex $V_{1\infty}$, whilst τ^* is an asymptotic plane;

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[1]) H. LIEBMANN, "Nichteuklidische Geometrie", Collection Schubert XLIX, page 63.

as base circle we can obtain a circle with finite radius in τ. For the generatrices of this cone the second point P_2 coincides with the second point at infinity; so the projection consists of the infinite segment $O_1DP'_{1\infty}$ and the isolated second point at infinity of this line.

For lines l outside this cone this isolated point vanishes, and on account of this the second vanishing point; for its determination remain however D, and the first vanishing point O_1. Now however, the perpendicular OS still increasing, l can become parallel to p_1, and hence parallel to e or to τ; it is then at right angles to the bisectrix of the acute angle between p_1 and OO_1, as well as to that of the right angle between e and $O_1V_{1\varkappa}$, which bisectrices are respectively divergent. All lines showing this property form a second asymptotic cone of revolution, for which however τ is now the asymptotic plane; they have a picture point at infinity, but are no less determined by this point and the first vanishing point O_1.

If l also lies outside this second cone, it becomes divergent with respect to τ, so it loses its picture point D; but now its second point at infinity lies again inside the cone $\varkappa$, which makes it projectible, so that in this case l has two vanishing points but no picture point; however, the two vanishing points are sufficient for its determination (see N°. 3). The originals at infinity corresponding to it are both under the picture plane; in connection with the preceding it would be preferable, in order to avoid confusion, to say that l has in this case two "first points at infinity" and therefore also two "first vanishing points".

The lines containing the second point at infinity $V_{2\infty}$ of OO_1 behave in like manner; we again find two asymptotic cones of revolution, one with the asymptotic plane τ, a second with the asymptotic plane τ^*, and we terminate with lines with two "second vanishing points" and without picture point.

Delft, September 1905.

Physiology. — "*A method for determining the osmotic pressure of very small quantities of liquid.*" By Prof. H. J. HAMBURGER.

It not unfrequently happens that one wishes to know the osmotic pressure of normal or pathological somatic fluids of which no more than $^1/_2$ or $^1/_4$ cc. are available. I recently had such a case when an oculist asked me what should be the concentration of liquids used for the treatment of the eye. It seemed to me to be rational — and the investigations of MASSART [1]) justified this opinion — to prescribe

[1]) MASSART, Archives de Biologie 9 1889, p. 335.

concentrations of the same osmotic pressure as the natural medium of the cornea and conjunctiva, namely the lachrymal fluid.

Until now, however, this pressure had not been measured, at any rate by a direct method, probably on account of the difficulty of obtaining a quantity of that fluid, sufficient for the customary methods, viz. the freezing-point and the blood corpuscles method.

So I tried to find a method with which $^1/_2$ cc., if necessary $^1/_4$ cc. of liquid should be sufficient. I succeeded in finding such a method.

It is based on the already known principle that the volume of blood corpuscles is greatly dependent on the osmotic pressure of the solution containing them. [1])

This principle has been applied here in the following manner. The fluid to be examined is put into a small, funnel-shaped glass tube, the cylindrical neck of which is formed by a calibrated capillary, closed below. [2]) Let this quantity be $^1/_2$ cc. Into other similar funnel-shaped tubes of the same size are put solutions of Na Cl of different concentration (0.8 %, 0.9 %, 1 %, 1.1 %, 1.2 %, 1.3 %, 1·4 %, 1.5 %, 1.6 %) and to each of these 0.02 cc. of blood is added. After half an hour — during which time the corpuscles are sure to have found osmotic equilibrium with their surroundings — the tubes are centrifuged until the sediments no longer alter their volumes. It is obvious that the osmotic pressure of the fluid under examination will be equal to that of the Na Cl solution in which the sediment of blood corpuscles is the same as in the fluid examined. We passingly remark that this solution of Na Cl in the case of lachrymal fluid contained 1.4 %.

A few remarks must be added.

Firstly it may be asked whether the blood which is added to the fluid to be examined does not appreciably alter the osmotic pressure of that fluid.

Assuming that the blood used contains 60 pCt. of serum, 0.02 cc. of blood will contain 0.012 cc. of serum. If the quantity of fluid was 1/2 cc., the total quantity of fluid will now be $0.012 + 0.5 = 0.512$ cc. If the fluid to be examined had an osmotic pressure of a 1.2 pCt. Na Cl solution and the serum that of a 0.9 pCt, Na Cl solution, dilution of the fluid with the serum will have produced a liquid with an osmotic pressure of $\frac{0.012 \times 0.9 + 0.5 \times 1.2}{0.012 + 0.5} = 1.19$ pCt. Na Cl. The osmotic pressure of the fluid is consequently reduced

1) Hamburger, Centralblatt f. Physiologie, 17 Juni 1893.

2) Hamburger Journal de Physiol. norm. et pathologique 1900 p. 889.

by 0.01 pCt Na Cl by the addition of 0.02 cc. of serum, a difference which cannot even be detected with the BECKMANN apparatus.

If instead of 1/2 cc. of fluid only 1/4 cc. has been used, a similar calculation shows that the osmotic pressure of the fluid under examination is diminished by a 0.014 pCt. Na Cl solution, corresponding to a depression of scarcely 0.0084°, a difference of depression, lying near the limit of accuracy of BECKMANN's determination of the freezing point.

However, if the difference were greater, this could be no objection to the method, since also the Na Cl solutions are mixed with the same quantity of blood.

The second remark concerns the pipette and the tubes.

In order to measure accurately, the bore of the pipette must be narrow and accordingly the instrument itself long. The column of 0.02 cc. of blood has a length of 143 mm. The same remark applies to the funnel-shaped tubes. The calibrated capillary part has a length of 57 millimetres and a volume of 0.01 cc. and is divided into 100 equal parts, which can easily be observed with the naked eye, fractions being estimated with a magnifying glass.

The use of funnel-shaped tubes of the same length, but with a still smaller volume of the capillary part than 0.01 cc., which would enable us to make determinations of the osmotic pressure of much smaller quantities of liquid than $^1/_4$ cc., would give rise to technical difficulties, on which I will not dwell here. No more shall I mention here the special precautions in experimenting, necessary for obtaining accurate figures. This subject will be dealt with elsewhere.

In order to give an idea of the reliability of the method, a table follows, containing two series of parallel experiments. (See p. 397).

The agreement between the figures (each division represents $\frac{0.01}{100} = 0.0001$ cc.) is seen to be very satisfactory.

A third remark concerns the possibility of making one or more additional determinations with the same fluid, for checking the result obtained. All one has to do is to drain off the liquid above the sediment by means of a finely drawn tube or pipette and to convey it into another funnel-shaped tube, to add again 0.02 cc. of blood and to centrifuge in the same way. The liquids in the NaCl-tubes are treated in the same way. Undoubtedly one changes the osmotic pressure of the liquids a little by again introducing 0.02 cc. of blood, but this is done with all the liquids and so the alteration

Salt solutions.	Volumes of the sediments after centrifuging for.				
	¼ hour.	½ hour.	¾ hour.	1 hour.	15 min.
Na Cl 0.9 %	74	69	68	68	68
» »	73	69	68	68	68
Na Cl 1.1 %	71	65	64	64	64
» »	68	64	64½	64½	64½
Na Cl 1.2 %	68	65	63	61½	61½
» »	69	65½	63	62	62
Na Cl 1.3 %	67	62	60	59	59
» »	67	62	60	59	59
Na Cl 1.4 %	69	62	58½	57	57
» »	67	62½	58	57½	57½
Na Cl 1.5 %	62	58	55	55	55
» »	64	59	56	56	56
lachrymal fluid	78	63	59½	57	57
the same lachr. fl.	76½	64	60	57	57

has no influence on the result, as would be the case if fresh solutions of NaCl were taken each time.

A last remark concerns the applicability of the method. It cannot be used so generally as the freezing point method: It cannot be applied with gall since this fluid contains substances causing haemolysis; it also fails for urine, since this fluid contains a relatively large quantity of urea which contributes considerably to the osmotic pressure but has no influence on the volume of the blood corpuscles.

For a number of other fluids, as blood serum, lymph, cerebrospinal fluid, saliva, lachrymal fluid, etc., the method can be successfully applied. It does not matter whether the fluids are coloured, for the determination only depends on the volume occupied in the fluid by the blood corpuscles.

Zoology. — *"On the primordial cranium of Tarsius spectrum"*. By Prof. Dr. EUGEN FISCHER of Freiburg i. B. (Preliminary paper). Communicated by Prof. A. A. W. HUBRECHT.

An investigation of the primordial cranium of Tarsius spectrum seemed particularly interesting to me as it might fill up a gap I had found when making a comparative study of the cartilaginous skull of apes and man on one hand, and of lower mammals (the mole) on the other [1]). So I was exceedingly happy when Prof. HUBRECHT,

[1]) FISCHER E. Das Primordialcranium von Talpa europae. Anat. Hefte Bd. 17, 1901 and: zur Entwicklungsgeschichte des Affenschädels. Zeitschr. f. Morph. und Anthrop. Bd. 5, 1903.

with generous kindness, placed at my disposal, out of his rich and valuable collection of embryos, such stages as were proper for this investigation.

In what follows a brief description will be given of the form and development of the chondrocranium as it appears at the height of development; this description is based on the reconstructed waxmodel which I made of the skull of an embryo of 34 mm. length. Other details of this embryo are shown in KEIBEL's Normentafel [1]).

Since an extensive and illustrated description will follow elsewhere, I shall be very brief here and give no detailed information as to literature and comparisons. For the first and also for the nomenclature used and the meaning of many only shortly mentioned details I refer to GAUPP's brilliant comparative of the development history of the vertebrate cranium in HERTWIG's Handbook [2]).

The basal plate is broad behind and well developed; anteriorly it delimits the foramen magnum. It is perforated by the hypoglossus. Laterally is has a fixed connection with the ear-capsule. This connection, however, is pierced by the narrow and long, almost slit-shaped foramen jugulare. Behind it, starting from the junction of the basal plate and the ear-capsule the cartilaginous plate develops which upwards represents the parietal plate, backwards and inwards the tectum synoticum. This tectum is a very narrow strap. So in this respect Tarsius resembles the young foetus of monkeys and man (cf. BOLK, Petr. Camp. II) and differs from the other mammals, where a broad plate is found.

Further forwards the basal plate itself becomes very remarkably narrow, so that here it consists only of a thin, round projection. At the same time it is separated by long slits from the two ear-capsules, with the anterior parts of which it only coalesces again in the region of the sella. This thin projection rises rather steeply, and in the sella region it becomes quite considerable with its two processus clinoidei posteriores. The two slits terminate close by, after having grown very narrow. Their existence seems to be very rare in mammals; they are defects which may be compared with the fenestra basicranialis posterior of Reptiles (GAUPP).

The ear-capsules themselves showed no peculiarities; they are

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[1]) HUBRECHT und KEIBEL, Normentafeln zur Entwicklung von Tarsius spectrum und Nycticebus tardigradus. Jena 1906. Tabelle N°. 36. Fig. 20 *a.—c.*

I am also greatly indebted to Prof. KEIBEL for enabling me to use the splendid series of sections of HUBRECHT's Tarsius embryos on which his own investigations were effectuated.

[2]) GAUPP. Die Entwicklung des Kopfskelettes. Hertwig's Handbuch 1905. Cap 6 p. 573.

moderately erect, the fossa subarcuata is only indicated. The foramen acusticum and higher upwards the foramen Nervi facialis mark the border of the vestibular and cochlear parts; to the former are attached above the parietal plates; they are very small and insignificant. A foramen jugulare spurium perforates its base. Frontally they send out a very short processus marginalis posterior, exactly as in the monkey skull. On the exterior of the ear-capsules lie, in exactly the same way as I described for the mole and embryos of apes, the cartilaginous stirrup, anvil and hammer, passing into MECKEL's cartilage.

The orbito-temporal part is characterised by its relatively phenomenal length. The continuation of the cranial trabecle from the saddle groove, where it had much broadened, is a narrow high ridge, a true septum interorbitale (GAUPP) still more extended than I found it with apes, although not so high as there. So the cranium is clearly tropibasical. By this long septum, which in front of course passes into the nasal septum, the nasal capsule is far separated from the brain capsule; it lies far in front of it, exactly as with Reptiles. The relatively large eyes of Tarsius are probably the cause of the survival of this extremely primitive formation.

Where the described cartilaginous beam broadens into the hypophysis groove it sends out, fairly deep towards the base, a round stalk at each side, bearing the small ala temporalis, which tapers in the same way as with the foetus of man and ape. It does not serve as a cranial wall yet, and has no Foramina rotunda and ovalia yet. Above it starts with two roots the large ala orbitalis. Between the roots the two foramina optica, the right and left one, are very close together, so that only the thin septum mentioned separates them. The orbital wings, themselves large plates, are neither bent upwards so strongly as with the lower mammals (the mole), nor do they extend laterally in such a perfect plane as with ape and man, but their shape is exactly between the two extremes, they slant sideways and upwards. Also the circumstance that their posterior end lengthens out into a real, although very thin taenia marginalis, which nearly reaches the parietal plate (there remains a very small gap), shows a similar transitional stage between the Primates and the other mammals. The anterior parts of the alae orbitales are not connected with the nasal capsule as usual (also in the sheep e. g. this connection is wanting according to DECKER). Below the sphenoid beam are, isolated from it, the roundish pterygoid cartilages, quite independent.

Proximally the septum interorbitale, as has been stated, passes

into the nasal septum. The nasal capsule has a remarkable resemblance with that of the Primates; there is no trace of the double tube form of other mammals.

The two apertures for the olfactory nerves are both simple, without any formation of cribrosa.

This part of the future nasal root is relatively broad, which is especially conspicuous with regard to the completed cranium.

Basally the whole nasal capsule has a slit-shaped opening, i. e. the bottom (lamina transversalis post. and ant.) is lacking, this being also characteristic for man and partly for apes, with whom I still found an indication of the bottom (Semnopithecus). About the yet slightly developed conchae, the cartilages of JACOBSON, the alar cartilage enclosing the nasal entrance, nothing particular can be mentioned.

MECKEL's cartilage proceeds well developed as far as the point of the chin and here has a continuous connection, without any trace of a suture, with that of the other side. REICHERT's cartilage proceeds continuously to the tongue bone.

On the dermal bones I will not dwell here; besides the upper squama of the occipital, resp. interparietal, all membrane-bones are present; the annulus tympanicus is only $^2/_3$ of a ring; frontal and parietal extend as yet to such a small height that the top of the skull is mostly covered with skin only.

When we now survey the whole cranium, as sketched above, we find two important characteristics. On one hand appears the exceedingly close relationship of the developing cranium of Tarsius and that of ape and man. In spite of clear specific peculiarities, it evidently stands much nearer to these than to the other known mammalian crania. This affords a new proof for the correctness of HUBRECHT's opinion as to the position of Tarsius in the system. At the same time an investigation of the primordial cranium of true Lemurs becomes necessary and promises important results. This investigation will shortly follow.

Secondly the resemblance between this type of skull and that of Reptiles is striking; like the skull of monkeys, so that of Tarsius in its cartilaginous stage, pleads unmistakably for unity of plan and origin of the Reptilian and Mammalian skull (cf. GAUPP's various articles). In our case the position of the nasal capsule, the septum interorbitale, a series of details in the arrangement of the foramina, the cartilaginous straps, etc. point clearly in that direction.

The study of each single form may in this way contribute to the solution of the problem of phylogenesis.

Freiburg i. B., October 1905.

Physics. — *"On the radiation of heat in a system of bodies having a uniform temperature"*. By Prof. H. A. LORENTZ.

(Communicated in the meetings of September and October 1905).

§ 1. A system of bodies surrounded by a perfectly black enclosure which is kept at a definite temperature, or by perfectly reflecting walls, will, in a longer or a shorter time, attain a state of equilibrium, in which each body loses as much heat by radiation as it gains by absorption, the intervening transparent media being the seat of an energy of radiation, whose amount per unit of volume is wholly determinate for every wave-length. The object of the following considerations is to examine somewhat more closely this state of things and to assign to each element of volume its part in the emission and the absorption. Of course, the most satisfactory way of doing this would be to develop a complete theory of the motions of electrons to which the phenomena may in all probability be ascribed. Unfortunately however, it seems very difficult to go as far as that. I have therefore thought it advisable to take another course, based on the conception of certain periodic electromotive forces acting in the elements of volume of ponderable bodies and producing the radiation that is emitted by these elements. If, without speaking of electrons, or even of molecules, we suppose such forces to exist in a matter continuously distributed in space, and if we suppose the emissivity of a black body to be known as a function of the temperature and the wave-length, we shall be able to calculate the intensity that must be assigned to the electromotive forces in question. The result will be a knowledge, not of the real mechanism of radiation, but of an imaginary one by which the same effects could be produced.

§ 2. For the sake of generality we shall consider a system of aeolotropic bodies. As to the notations used in our equations and the units in which the electromagnetic quantities are expressed, these will be the same that I have used in my articles in the Mathematical Encyclopedia. We may therefore start from the following general relations between the electric force $\mathfrak{E}$, the current $\mathfrak{C}$, the magnetic force $\mathfrak{H}$ and the magnetic induction $\mathfrak{B}$

$$rot\ \mathfrak{H} = \frac{1}{c}\mathfrak{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$rot\ \mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{B}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In these formulae c denotes the velocity of light in the aether.

In the greater part of what follows, we shall confine ourselves to cases, in which the components of the above vectors and of others we shall have occasion to consider, are harmonic functions of the time with the frequency n. Then, the mathematical calculations can be much simplified if, instead of the real values of these components, we introduce certain complex quantities, all of which contain the time in the factor e^{int} and whose real parts are the values of the components with which we are concerned. If $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ are complex quantities of this kind, relating in one way or another to the three axes of coordinates and in which the quantity e^{int} may be multiplied by complex quantities, the combination $(\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z)$ may be called a complex vector $\mathfrak{A}$ and $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ its components.

By the real part of such a vector we shall understand a vector whose components are the real parts of $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$. It will lead to no confusion, if the same symbol is used alternately to denote a complex vector and its real part. It will also be found convenient to speak of the rotation and the divergence of a complex vector, and of the scalar product $(\mathfrak{A}, \mathfrak{B})$ and the vector product $[\mathfrak{A}.\mathfrak{B}]$ of two complex vectors $\mathfrak{A}$ and $\mathfrak{B}$, all these quantities being defined in the same way as the corresponding ones in the case of real vectors. E. g., we shall mean by the scalar product $(\mathfrak{A}.\ \mathfrak{B})$ the expression $\mathfrak{A}_x\mathfrak{B}_x + \mathfrak{A}_y\mathfrak{B}_y + \mathfrak{A}_z\mathfrak{B}_z$.

It is easily seen that, if $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{C}$ and $\mathfrak{B}$ are complex vectors, satisfying the equations (1) and (2), their real parts will do so likewise. The denominations electric force, etc. will be applied to these complex vectors as well as to the real ones.

One advantage that is gained by the use of complex quantities lies in the fact that now, owing to the factor e^{int}, a differentiation with respect to the time amounts to the same thing as a multiplication by in; in virtue of this the relation between $\mathfrak{E}$ and $\mathfrak{C}$ and that between $\mathfrak{H}$ and $\mathfrak{B}$ may be expressed in a simple form. Indeed, we may safely assume that, whatever be the peculiar properties of a ponderable body, the components of $\mathfrak{C}$ are connected to those of $\mathfrak{E}$ by three linear equations with constant coefficients, containing the components and their differential coefficients with respect to the time. In the case of the complex vectors, these equations may be written as linear relations between the components themselves; in other terms, one complex vector becomes a linear vector function of the other. A relation of this kind between two vectors $\mathfrak{A}$ and $\mathfrak{B}$ can always be expressed by three equations of the form

$$\mathfrak{B}_x = v_{11}\,\mathfrak{A}_x + v_{12}\,\mathfrak{A}_y + v_{13}\,\mathfrak{A}_z,$$
$$\mathfrak{B}_y = v_{21}\,\mathfrak{A}_x + v_{22}\,\mathfrak{A}_y + v_{23}\,\mathfrak{A}_z,$$
$$\mathfrak{B}_z = v_{31}\,\mathfrak{A}_x + v_{32}\,\mathfrak{A}_y + v_{33}\,\mathfrak{A}_z,$$

which we shall condense into the formula

$$\mathfrak{B} = (v)\,\mathfrak{A}.$$

According to this notation we may put $\mathfrak{C} = (p')\,\mathfrak{E}$, or, as is more convenient for our purpose,

$$\mathfrak{E} = (p)\,\mathfrak{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the symbol (p) containing a certain number of coefficients p which are determined by the properties of the body considered. As a rule, these coefficients are complex quantities, whose values depend on the frequency n.

As to the relation between $\mathfrak{B}$ and $\mathfrak{H}$, we shall put

$$\mathfrak{B} = (\mu)\,\mathfrak{H},$$

or

$$\mathfrak{H} = (q)\,\mathfrak{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have further to introduce an electromotive force which will be represented by a vector $\mathfrak{E}_e$, or by the real part of a complex vector $\mathfrak{E}_e$. The meaning of this is simply that the current $\mathfrak{C}$ is supposed to depend on the vector $\mathfrak{E} + \mathfrak{E}_e$ in the same way in which it depends on $\mathfrak{E}$ alone in ordinary cases, so that

$$\mathfrak{E} + \mathfrak{E}_e = (p)\,\mathfrak{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, we may assume a magnetomotive force $\mathfrak{H}_e$, replacing (4) by

$$\mathfrak{H} + \mathfrak{H}_e = (q)\,\mathfrak{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This new vector $\mathfrak{H}_e$ however, does not correspond to any really existing quantity; it is only introduced for the purpose of simplifying the demonstration of a certain theorem we shall have to use.

As to the coefficients we have taken together in the symbols (p) and (q), we shall suppose them to be connected with each other in the way expressed by

$$p_{12} = p_{21},\ p_{23} = p_{32},\ p_{31} = p_{13}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$q_{12} = q_{21},\ q_{23} = q_{32},\ q_{31} = q_{13} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The only case excluded by this assumption is that of a body placed in a magnetic field.

For isotropic bodies we may write, instead of (5) and (6),

$$\mathfrak{E} + \mathfrak{E}_e = p\,\mathfrak{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$
$$\mathfrak{H} + \mathfrak{H}_e = q\,\mathfrak{B}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

with only one complex coefficient p and one coefficient q.

§ 3. Before coming to the problem we have in view, it is necessary to treat some preliminary questions. In the first place, we shall examine the vibrations that are set up in an unlimited homogeneous and isotropic body subjected to given electromotive and magnetomotive forces, changing with the frequency n. This problem is best treated by using the complex vectors.

We may deduce from (1)

$$rot\ rot\ \mathfrak{H} = \frac{1}{c}\, rot\, \mathfrak{E},$$

or

$$grad\ div\ \mathfrak{H} - \Delta\, \mathfrak{H} = \frac{1}{c}\, rot\, \mathfrak{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and similarly from (2)

$$grad\ div\ \mathfrak{E} - \Delta\, \mathfrak{E} = -\frac{1}{c}\, rot\, \mathfrak{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Again, always using the equations (1), (2), (9) and (10), we find

$$div\ \mathfrak{B} = 0 \quad , \quad div\ \mathfrak{E} = 0,$$

$$div\ \mathfrak{H} = -\, div\, \mathfrak{H}_e, \quad div\ \mathfrak{E} = -\, div\, \mathfrak{E}_e,$$

$$rot\ \mathfrak{E} = \frac{1}{p}(rot\ \mathfrak{E} + rot\ \mathfrak{E}_e') = -\frac{1}{pc}\dot{\mathfrak{B}} + \frac{1}{p}\, rot\, \mathfrak{E}_e$$

$$= -\frac{1}{pqc}(\dot{\mathfrak{H}} + \dot{\mathfrak{H}}_e) + \frac{1}{p}\, rot\, \mathfrak{E}_e,$$

$$rot\ \dot{\mathfrak{B}} = \frac{1}{q}(rot\, \dot{\mathfrak{H}} + rot\, \dot{\mathfrak{H}}_e) = \frac{1}{qc}\dot{\mathfrak{E}} + \frac{1}{q}\, rot\, \dot{\mathfrak{E}}_e$$

$$= \frac{1}{pqc}(\dot{\mathfrak{E}} + \dot{\mathfrak{E}}_e) + \frac{1}{q} rot\, \dot{\mathfrak{H}}_e,$$

so that (11) and (12) become

$$\Delta\, \mathfrak{H} - \frac{1}{pqc^2}\dot{\mathfrak{H}} = -\, grad\ div\ \mathfrak{H}_e + \frac{1}{pqc^2}\dot{\mathfrak{H}}_e - \frac{1}{pc} rot\, \mathfrak{E}_e,$$

$$\Delta\, \mathfrak{E} - \frac{1}{pqc^2}\dot{\mathfrak{E}} = -\, grad\ div\ \mathfrak{E}_e + \frac{1}{pqc^2}\dot{\mathfrak{E}}_e + \frac{1}{qc}\, rot\, \dot{\mathfrak{H}}_e.$$

The solution of these equations may be put in a convenient form by means of two auxiliary vectors $\mathfrak{A}$ and $\mathfrak{Q}$. If these are determined by

$$\Delta\, \mathfrak{A} - \frac{1}{pqc^2}\dot{\mathfrak{A}} = -\, \mathfrak{E}_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\Delta\, \mathfrak{Q} - \frac{1}{pqc}\dot{\mathfrak{Q}} = -\, \mathfrak{H}_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we shall have

$$\mathfrak{H} = grad\ div\ \mathfrak{Q} - \frac{1}{pqc^2}\dot{\mathfrak{Q}} + \frac{1}{pc}\, rot\, \mathfrak{A}, \quad . \quad . \quad . \quad . \quad (15)$$

$$\mathfrak{E} = grad\ div\ \mathfrak{A} - \frac{1}{pqc^2}\dot{\mathfrak{A}} - \frac{1}{qc}\,rot\,\dot{\Omega} \quad . \quad . \quad . \quad . \quad (16)$$

Finally, putting

$$v^2 = i\,p\,q\,n\,c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

we get instead of (13) and (14)

$$\Delta\,\mathfrak{A} - \frac{1}{v^2}\ddot{\mathfrak{A}} = -\,\mathfrak{E}_e\,,$$

$$\Delta\,\Omega - \frac{1}{v^2}\ddot{\Omega} = -\,\mathfrak{H}_e,$$

for the solution of which we may take

$$\mathfrak{A} = \frac{1}{4\pi}\int \frac{1}{r}\,\mathfrak{E}_e\left(t - \frac{r}{v}\right) d\,S, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\Omega = \frac{1}{4\pi}\int \frac{1}{r}\,\mathfrak{H}_e\left(t - \frac{r}{v}\right) d\,S. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Here $d\,S$ denotes an element of volume situated at a distance r from the point for which we wish to calculate $\mathfrak{A}$ and Ω, and the index $\left(t - \frac{r}{v}\right)$ means that, in the expressions representing $\mathfrak{E}_e$ and $\mathfrak{H}_e$ for that element of volume, t is to be replaced by $t - \frac{r}{v}$.

The algebraic sign of v is left indeterminate by (17). We shall choose it in such a way that our formulae represent a propagation of vibrations *issuing* from the elements of volume in which $\mathfrak{E}_e$ and $\mathfrak{H}_e$ are applied.

For aether we have $q = 1$ and, as may easily be shown $\frac{1}{p} = i\,n$, $v = c$.

§ 4. We have next to establish the equation of energy. The calculations required for this purpose, as well as those we shall have to perform later on, may be much simplified, if we replace all discontinuities at the limit of two bodies by a gradual transition from one to the other; this may be done without loss of generality, because, in our final results, the thickness of the boundary layers may be made to become infinitely small. A further simplification is obtained by leaving out of consideration the imaginary magnetomotive forces, and by supposing the coefficients μ and q to be real. The coefficients p_{11}, p_{12} etc. however will always be considered as complex quantities. We shall decompose them into their real parts, which we shall denote by α_{11}, α_{12}, etc., and their imaginary parts, for which we shall write $-\,i\,\beta_{11}, -\,i\,\beta_{12}$, etc., so that $p_{11} = \alpha_{11} - i\,\beta_{11}$, etc.

The equation (5) now becomes

$$\mathfrak{E} + \mathfrak{E}_e = (\alpha)\,\mathfrak{C} - i\,(\beta)\,\mathfrak{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or, if we define a new vector $\mathfrak{D}$ by means of the equation

$$\mathfrak{C} = \dot{\mathfrak{D}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\mathfrak{E} + \mathfrak{E}_e = (\alpha)\,\mathfrak{C} + n\,(\beta)\,\mathfrak{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the deduction of the equation of energy we have to understand by $\mathfrak{E}$, $\mathfrak{E}_e$, $\mathfrak{H}$ and $\mathfrak{D}$ the real vectors. For these we have the formulae (1), (2) and (21), and besides, since q, α and β are real, the relations (4) and (22).

From (1) and (2) we may draw immediately

$$c\,\{(\mathfrak{H}\,.\,rot\,\mathfrak{E}) - (\mathfrak{E}\,.\,rot\,\mathfrak{H})\} = -(\mathfrak{H}\,.\,\dot{\mathfrak{B}}) - (\mathfrak{E}\,.\,\mathfrak{C}),$$

the left-hand member of which is

$$div\,\mathfrak{S},$$

if we define the vector $\mathfrak{S}$ by the equation

$$\mathfrak{S} = c\,[\mathfrak{E}\,.\,\mathfrak{H}], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

i.e., if we understand by it the vector product of $\mathfrak{E}$ and $\mathfrak{H}$, multiplied by c.

In the right-hand member we have in the first place

$$(\mathfrak{H}\,.\,\dot{\mathfrak{B}}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathfrak{H}\,.\,\mathfrak{B}),$$

as may be seen from (4), if (8) is taken into account, and further, in virtue of (7), (21) and (22),

$$(\mathfrak{E}\,.\,\mathfrak{C}) = ((\alpha)\,\mathfrak{C}\,.\,\mathfrak{C}) + \frac{1}{2}n\frac{\partial}{\partial t}((\beta)\,\mathfrak{D}\,.\,\mathfrak{D}) - (\mathfrak{E}_e\,.\,\mathfrak{C}).$$

Our equation therefore takes the following form, in which the meaning of the different terms is at once apparent,

$$(\mathfrak{E}_e\,.\,\mathfrak{C}) = ((\alpha)\,\mathfrak{C}\,.\,\mathfrak{C}) + \frac{1}{2}n\frac{\partial}{\partial t}((\beta)\,\mathfrak{D}\,.\,\mathfrak{D}) + \frac{1}{2}\frac{\partial}{\partial t}(\mathfrak{H}\,.\,\mathfrak{B}) + div\,\mathfrak{S}.$$

The first member represents the work done by the electromotive force per unit of volume and unit of time; in the second member

$$w = ((\alpha)\,\mathfrak{C}\,.\,\mathfrak{C}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

is the expression for the quantity of heat that is developed per unit of space and unit of time. Further, $\frac{1}{2}(\mathfrak{H}\,.\,\mathfrak{B})$ is the magnetic and $\frac{1}{2}n\,((\beta)\,\mathfrak{D}\,.\,\mathfrak{D})$ the electric energy, both reckoned per unit of volume. The vector $\mathfrak{S}$ denotes the flow of energy, so that the amount of energy an element of volume dS loses by this flow is given by $div\,\mathfrak{S}\,dS$.

§ 5. We may now pass to a theorem which I have formerly proved in a somewhat more cumbrous and less general way. In order to arrive at it, we have to use the complex vectors, supposing at the same time the existence of magnetomotive forces; we have therefore to apply the formulae (5) and (6).

We shall consider *two* different states with the same frequency n, both of which can exist in the system of bodies. The symbols $\mathfrak{E}$, $\mathfrak{H}$, etc. will be used for one state and the corresponding symbols, distinguished by accents, for the other. We shall proceed in a way much like the operations of the last paragraph, with this difference however, that we shall now combine quantities relating to one state with quantities belonging to the other.

We shall start from the relation

$$c\{(\mathfrak{H}'.\ rot\ \mathfrak{E}) - (\mathfrak{E}.\ rot\ \mathfrak{H}')\} = -(\dot{\mathfrak{H}}'.\ \mathfrak{B}) - (\mathfrak{E}.\ \dot{\mathfrak{C}}').$$

Here the expression on the left is equal to

$$c\ div\ [\mathfrak{E}.\ \mathfrak{H}']$$

and on the other side we may put

$$(\mathfrak{H}'.\ \dot{\mathfrak{B}}) = in\,(\mathfrak{H}'.\ \mathfrak{B}) = in\,((q)\,\mathfrak{B}'.\ \mathfrak{B}) - (\mathfrak{H}'_e.\ \dot{\mathfrak{B}}),$$
$$(\mathfrak{E}.\ \dot{\mathfrak{C}}') = ((p)\,\mathfrak{C}.\ \dot{\mathfrak{C}}') - (\mathfrak{E}_e.\ \dot{\mathfrak{C}}'),$$

so that we find

$$c\ div\ [\mathfrak{E}.\ \mathfrak{H}'] = -in\,((q)\,\mathfrak{B}'.\ \mathfrak{B}) - ((p)\mathfrak{C}\ .\ \dot{\mathfrak{C}}') + (\mathfrak{H}'_e.\ \dot{\mathfrak{B}}) + (\mathfrak{E}_e.\ \dot{\mathfrak{C}}').$$

The theorem in question is a consequence of this formula and the corresponding one that is got by interchanging the quantities belonging to the two states; we have only to subtract one equation from the other. Since, by (8) and (7)

$$((q)\,\mathfrak{B}'.\ \mathfrak{B}) = ((q)\,\mathfrak{B}.\ \mathfrak{B}') \text{ and } ((p)\,\mathfrak{C}.\ \mathfrak{C}') = ((p)\,\mathfrak{C}'.\ \mathfrak{C}),$$

we find in this way

$$c\{div\,[\mathfrak{E}.\ \mathfrak{H}'] - div\,[\mathfrak{E}'.\ \mathfrak{H}]\} = (\dot{\mathfrak{H}}'_e\ \mathfrak{B}) - (\mathfrak{H}_e.\ \dot{\mathfrak{B}}') + (\mathfrak{E}_e.\ \dot{\mathfrak{C}}') - (\mathfrak{E}'_e.\ \mathfrak{C}).$$

We shall finally multiply this by an element of volume dS, and take the integral of both sides over the space within a closed surface σ. If we denote by n the normal to the latter, drawn outward, the result will be

$$c\int\{[\mathfrak{E}.\ \mathfrak{H}']_n - [\mathfrak{E}'.\ \mathfrak{H}]_n\}\,d\sigma = \int\{(\dot{\mathfrak{H}}'_e.\mathfrak{B}) - (\mathfrak{H}_e.\dot{\mathfrak{B}}') + (\mathfrak{E}_e.\dot{\mathfrak{C}}') - (\dot{\mathfrak{C}}'_e.\mathfrak{C})\}\,dS \quad (25)$$

§ 6. There are a number of cases in which the first member of this equation is zero.

a. E. g. we may suppose the system to be limited on all sides in such a way that it cannot exchange rays with surrounding bodies; we can realize this by enclosing the system in an envelop that is

perfectly reflecting on the outside. If, under these circumstances, the surface σ surrounds that envelop, we may put in every point of it $\mathfrak{E} = 0, \mathfrak{E}' = 0, \mathfrak{H} = 0, \mathfrak{H}' = 0$.

b. If the envelop is made of a perfectly conducting material, both the electric force $\mathfrak{E}$ and the force $\mathfrak{E}'$ will be normally directed in every point of its inner surface. Consequently, if the latter is chosen for the surface σ, we shall have

$$[\mathfrak{E} \,.\, \mathfrak{H}']_n = 0 \text{ and } [\mathfrak{E}' \,.\, \mathfrak{H}]_n = 0.$$

c. Finally we may conceive a system lying in a finite part of space and surrounded by aether, into which it emits rays travelling outwards to infinite distance. Taking in this case for σ a sphere of infinite radius, we shall show that for each element $d\sigma$ the factor by which it is multiplied in the equation (25) vanishes. The direction of the axes of coordinates being indeterminate, it will suffice to prove this proposition for the point P in which the sphere is cut by a line drawn from the centre O in the direction of the axis of x.

Now, if we confine ourselves to those parts of $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{E}'$ and $\mathfrak{H}'$ which are inversely proportional to the first power of OP, as may obviously be done, we may consider the state of things near the point P as a propagation of vibrations in the direction OP, the electric and magnetic force being perpendicular to that direction and to each other. Denoting by a and b, a' and b' certain complex quantities, we may write

$$\begin{aligned}
&\mathfrak{E}_x = 0, \mathfrak{E}_y = ae^{int}, && \mathfrak{E}_z = be^{int},\\
&\mathfrak{H}_x = 0, \mathfrak{H}_y = -be^{int}, && \mathfrak{H}_z = ae^{int},\\
&\mathfrak{E}'_x = 0, \mathfrak{E}'_y = a'e^{int}, && \mathfrak{E}'_z = b'e^{int},\\
&\mathfrak{H}'_x = 0, \mathfrak{H}'_y = -b'e^{int}, && \mathfrak{H}'_z = a'e^{int},
\end{aligned}$$

and we have at the point P, since in it the normal to the spherical surface is parallel to the axis of x,

$$[\mathfrak{E} \,.\, \mathfrak{H}']_n - [\mathfrak{E}' \,.\, \mathfrak{H}]_n = (\mathfrak{E}_y \,\mathfrak{H}'_z - \mathfrak{E}_z \,\mathfrak{H}'_y) - (\mathfrak{E}'_y \,\mathfrak{H}_z - \mathfrak{E}'_z \,\mathfrak{H}_y) = 0.$$

These considerations show that in many cases the equation (25) reduces to

$$\int \{(\mathfrak{E}'_e \,.\, \mathfrak{C}) - (\mathfrak{H}'_e \,.\, \dot{\mathfrak{B}})\}\, dS = \int \{(\mathfrak{E}_e \,.\, \mathfrak{C}') - (\mathfrak{H}_e \,.\, \dot{\mathfrak{B}}')\} dS \quad . \quad (26)$$

§ 7. It is particularly interesting to examine the effects produced by an electromotive or a magnetomotive force which is confined to an infinitely small space S. Let P be any point of this region, $\mathfrak{a}$ a real vector having everywhere the same direction h and the same magnitude $|\mathfrak{a}|$, and let us apply in all points of S an electromotive force $\mathfrak{a}\, e^{int}$. Then we shall say that there is an "electromotive

action" at the point P in the direction h. We may represent it by the symbol

$$\mathfrak{a}\, S\, e^{int}$$

and we may consider its intensity and its phase to be determined by the real part of $|\mathfrak{a}|\, S\, e^{int}$.

In a similar sense we can also conceive a "magnetomotive action" existing in some point of the system.

These definitions being agreed upon, equation (26) leads to the following remarkable conclusions.

a. Let there be, in the first of the two cases we have distinguished in the preceding paragraph, an electromotive action $\mathfrak{a}\, S\, e^{int}$ at the point P in the direction h, and in the second case an electromotive action $\mathfrak{a}'\, S' e^{int}$ at the point P' in the direction h', there being in neither case a magnetomotive force. Then the integrals in (26) are to be extended to the infinitely small spaces S′ and S and the result may be written in the form

$$(\mathfrak{a}' \,.\, \mathfrak{C}_{P'})\, S' = (\mathfrak{a} \,.\, \mathfrak{C}'_P)\, S\,,$$

if we represent by $\mathfrak{C}_{P'}$ the current produced in P' in the first case and by $\mathfrak{C}'_P$ the current existing in P in the second.

Hence, assuming the equality

$$|\,\mathfrak{a}\,|\, S = |\,\mathfrak{a}'\,|\, S'\,,$$

we conclude that

$$\mathfrak{C}_{h'P'} = \mathfrak{C}'_{hP} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The full meaning of this appears, if we write the two quantities in the form

$$\mathfrak{C}_{h'P'} = \mu\, e^{i(nt+\nu)} \quad \text{and} \quad \mathfrak{C}'_{hP} = \mu'\, e^{i(nt+\nu')}.$$

Indeed, (27) requires that

$$\mu = \mu'\,,\ \nu = \nu'\,,$$

and we have the theorem:

If an electromotive action applied at a point P in the direction h produces in a point P' a current whose component in an arbitrarily chosen direction h' has the amplitude μ and the phase ν, an equal electromotive action taking place at the point P' in the direction h' will produce a current in P, whose component in the direction h has exactly the same amplitude μ and the same phase ν.

b. Without changing anything in the circumstances of the first case, we shall now assume, that in the second the vibrations are excited not by electromotive forces, but by a magnetomotive action $\mathfrak{a}'\, S'\, e^{int}$, at the point P' in the direction h'. We then find

$$-(\mathfrak{a}' \,.\, \dot{\mathfrak{B}}_{P'})\, S' = (\mathfrak{a} \,.\, \mathfrak{C}'_P)\, S,$$

and, if we put

$$|\mathfrak{a}|\,\mathfrak{S} = |\mathfrak{a}'|\,\mathfrak{S}',$$

$$-\mathfrak{H}_{K'P'} = \mathfrak{C}'_{hP}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

a theorem similar to the former.

§ 8. The absorption of rays being measured by the amount of heat developed, the expression (24), in which $\mathfrak{C}$ is the real current, will be often used in what follows. It may be replaced by

$$w = (\mathfrak{F} \cdot \mathfrak{C}),$$

if we write $\mathfrak{F}$ for the vector $(\alpha)\,\mathfrak{C}$, so that

$$\mathfrak{F}_x = \alpha_{11}\,\mathfrak{C}_x + \alpha_{12}\,\mathfrak{C}_y + \alpha_{13}\,\mathfrak{C}_z, \text{ etc.} \quad . \quad . \quad . \quad . \quad (29)$$

Now, by a well known theorem, the axes of coordinates may always be chosen in such a way that the coefficients α_{12}, α_{23}, α_{31} in these equations become zero. Denoting the remaining coefficients by α_1, α_2, α_3, we have for the relation between $\mathfrak{F}$ and $\mathfrak{C}$

$$\mathfrak{F}_x = \alpha_1\,\mathfrak{C}_x, \quad \mathfrak{F}_y = \alpha_2\,\mathfrak{C}_y, \quad \mathfrak{F}_z = \alpha_3\,\mathfrak{C}_z,$$

and for the development of heat

$$w = \alpha_1\,\mathfrak{C}_x^2 + \alpha_2\,\mathfrak{C}_y^2 + \alpha_3\,\mathfrak{C}_z^2 . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The directions we must give to the axes in order to obtain these simplifications, may properly be called the *principal directions;* in general, they will not be the same for different frequencies. This is due to the fact that the coefficients in (29) depend on the value of n.

It is also to be noticed that by this choice of the axes of coordinates, the coefficients β_{12}, β_{23}, β_{31}, and p_{12}, p_{23}, p_{31} will not, in general, be made to become zero.

In the case of an isotropic body we may take as principal directions any three directions perpendicular to each other.

§ 9. Thus far we have only prepared ourselves for our main problem. In the next paragraphs we shall first consider the absorption by a very thin plate surrounded by aether on both sides, and receiving in the normal direction a beam of rays. Combining the result with the ratio between the emissivity and the coefficient of absorption of a body, we shall be able to determine the amount of energy, radiated by the plate in a normal direction, and our next object will be to calculate the intensity we must ascribe to electromotive forces acting in the plate (§ 1), in order to account for that radiation. This will lead us to a general hypothesis concerning the electromotive forces acting in the elements of volume of a ponderable body and we shall conclude by showing that, if these electromotive forces were applied, the condition required for the equilibrium of radiation would always be fulfilled.

§ 10. Let the plate be homogeneous, with its faces parallel to the first and the second principal direction. We shall take these for the axes of x and y, placing the origin O in the front surface of the plate, i. e. in the surface exposed to the rays, and drawing the axis of z toward the outside. As has already been said, the absorption will be calculated by means of the formula (30); it will therefore be determined by the components of $\mathfrak{C}$ and by those of $\mathfrak{E}$, on which they depend. Now, our problem is greatly simplified, if we suppose the thickness Δ of the plate to be infinitely small and if, in calculating the absorption, we confine ourselves to quantities of the first order of magnitude with respect to Δ. The quantity w relating to unit volume, we may then neglect all infinitely small terms in $\mathfrak{C}$ and $\mathfrak{E}$; consequently, we need not attend to the changes of these vectors in the plate along a line perpendicular to its faces. Moreover, in virtue of the well known conditions of continuity, the values of $\mathfrak{E}_x$ and $\mathfrak{E}_y$ within the plate will be equal to those existing in the aether immediately before it; also, $\mathfrak{C}_z$ will be 0, because it is so in the aether. For $\mathfrak{E}_x$ and $\mathfrak{E}_y$ we may even take the values, existing in the incident beam, the reason for this being that the values belonging to the reflected rays, (the vibrations reflected at the two sides being taken together) are proportional to the thickness, if the plate is infinitely thin.

It is seen by these considerations that in the case of a given incident motion, $\mathfrak{C}_x$, $\mathfrak{C}_y$, $\mathfrak{E}_z$ are the only unknown quantities in the three equations connecting the components of $\mathfrak{E}$ and $\mathfrak{C}$. We need not, however, work out the solution of these equations.

Finally, it must be kept in mind that, in the case of harmonic vibrations, the mean value of w for a lapse of time comprising many periods is given by

$$w = \frac{1}{2}\{a_1(\mathfrak{C}_x)^2 + a_2(\mathfrak{C}_y)^2 + a_3(\mathfrak{C}_z)^2\}, \quad . \quad . \quad . \quad (31)$$

if $(\mathfrak{C}_x)$, $(\mathfrak{C}_y)$, $(\mathfrak{C}_z)$ are the *amplitudes* of the components of the current.

§ 11. We shall in the first place assume that in the incident rays the electric force is parallel to the axis of x. Let its amplitude be a. Then, an element ω of the front surface will receive an amount of energy

$$\frac{1}{2}ca^2\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

per unit of time.

Within the plate, there will be electric currents in the directions of x and y. These will have amplitudes proportional to a, and for which we may therefore write:

$$(\mathfrak{E}_x) = f\,a \quad , \quad (\mathfrak{E}_y) = g\,a$$

denoting by f and g two factors, which it will be unnecessary to calculate. From (31) we deduce for the heat developed in the part $\omega\,\Delta$ of the plate,

$$\frac{1}{2}(\alpha_1 f^2 + \alpha_2 g^2)\,a^2\,\omega\,\Delta$$

and, dividing this by (32), for the coefficient of absorption

$$A = \frac{1}{c}(\alpha_1 f^2 + \alpha_2 g^2)\,\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Our next step must be to obtain a formula for the emission. For this purpose we fix our attention on a surface-element ω' parallel to the plate and situated at a large distance r from it, at a point of OZ. The electric vibrations issuing from the plate may be decomposed in the first place into vibrations of different frequencies and in the second place into components parallel to OX and OY.

After having effected this decomposition, we may attend to the amount of energy travelling across ω' per unit of time, in so far as it belongs to vibrations having the first of the two directions and to frequencies lying between the limits n and $n + dn$. Now, if the plate were removed, and if instead of it a perfectly black body of the same temperature were placed behind an opaque screen with an opening coinciding with the element ω, the radiation might be represented by

$$\frac{k\,\omega\,\omega'\,dn}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

an expression which may also be regarded as indicating the ratio between the emissivity of a body of any kind under the said circumstances and its coefficient of absorption. The experimental investigations of these last years have led to a knowledge of the coefficient k for a wide range of temperatures and frequencies.

By KIRCHHOFF's law, the flow of energy across the element ω', originated by the part

$$\omega\,\Delta = S$$

of the plate, in so far as it is due to vibrations of the said direction and frequency, is found by multiplying (34) by (33). Its amount is therefore

$$\frac{k\,S\,(\alpha_1 f^2 + \alpha_2 g^2)\,\omega'\,dn}{c\,r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and we have now to account for this radiation by means of suitable electromotive forces applied to the plate.

§ 12. We shall first put the question what must be the amplitude a_1 of an electromotive force acting in the direction of OX with the frequency n, if this force is to produce, on account of the electric vibrations parallel to OX, a flow of energy

$$\frac{k\,S\,a_1\,f^2\,\omega'\,dn}{c\,r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

across the element ω' at the point P. Since this flow may be represented by

$$\frac{1}{2}\,c\,b^2\,\omega',$$

if b is the amplitude of $\mathfrak{E}_x$ at the point P, we must have

$$b = \frac{f}{cr}\sqrt{2k\,S\,a_1\,dn}.$$

The amplitude of the current $\mathfrak{C}_x = \dot{\mathfrak{E}}_x$ must therefore be

$$\frac{nf}{cr}\sqrt{2k\,S\,a_1\,dn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

At this stage of our reasoning we may avail ourselves of the theorem of § 7, *a*. Indeed, if the electromotive force $\mathfrak{E}_{ex}$ in the part S of the plate must have the amplitude a_1 in order to call forth at the point P a current $\mathfrak{C}_x$ whose amplitude has the value (37), a_1 will also be the amplitude we must give to an electromotive force $\mathfrak{E}_{ex}$, acting in an element of volume S of the aether near P, if we wish to bring about by its action a current with the amplitude (37) in the plate. This is the condition by which we shall determine the value of a_1.

§ 13. The solution is readily obtained by means of the formulae (18) and (16). If, in an element of volume S of the aether, $\mathfrak{E}_{ex} = a_1\,e^{int}$, $\mathfrak{E}_{ey} = 0$, $\mathfrak{E}_{ez} = 0$, we shall have

$$\mathfrak{A}_x = \frac{a_1\,S}{4\,\pi\,r}\,e^{in\left(t-\frac{r}{c}\right)},\quad \mathfrak{A}_y = 0,\ \mathfrak{A}_z = 0,\ \mathfrak{Q} = 0$$

and

$$\mathfrak{E}_x = \frac{\partial^2\,\mathfrak{A}_x}{\partial\,x^2} + \frac{n^2}{c^2}\mathfrak{A}_x\,,$$

as may be easily seen, if the equations

$$p = \frac{1}{in},\quad q = 1,\quad \dot{\mathfrak{A}}_x = in\,\mathfrak{A}_x$$

are taken into account.

In the differential coefficients of $\mathfrak{A}_x$ we may omit all terms containing the square and higher powers of $\frac{1}{r}$. Hence, in a point of the

axis of z, which passes through the point P, $\frac{\partial^2 \mathfrak{A}_y}{\partial x^2} = 0$.

In this way, the electric force in the aether immediately before the plate is found to be

$$\mathfrak{E}_x = \frac{a_1 n^2 S}{4 \pi c^2 r} e^{in\left(t - \frac{r}{c}\right)}.$$

Its amplitude is

$$\frac{a_1 n^2 S}{4 \pi c^2 r} \quad \ldots \ldots \ldots \quad (38)$$

and that of the current $\mathfrak{C}_x$ within the plate

$$\frac{a_1 n^2 S f}{4 \pi c^2 r}.$$

This must be equal to the expression (37). The solution of our problem is therefore

$$a_1 = \frac{4 \pi c}{n} \sqrt{\frac{2 k a_1 dn}{S}} \quad \ldots \ldots \quad (39)$$

In the preceding formulae S means the volume of the portion of the plate we have considered. Now, after having decomposed this portion into a large number of elements of volume **s**, we may bring about just the same radiation by applying in each of these an electromotive force in the direction of OX with the amplitude

$$a_1 = \frac{4 \pi c}{n} \sqrt{\frac{2 k a_1 dn}{\mathbf{s}}}, \quad \ldots \ldots \quad (40)$$

provided only we suppose the electromotive forces in all these elements **s** to be independent of each other, so that their phases are distributed at random over the elements.

Indeed, from the fact that the force whose amplitude is (39), acting in the space S, gives rise to a radiation represented by (36), we may conclude that an electromotive force with the amplitude (40), when applied to the element **s**, will produce a flow of energy

$$\frac{k \mathbf{s} a_1 f^2 \omega' dn}{c r^2}$$

across the element ω'. A similar expression holds for each element **s** and, on account of the circumstance that the vibrations due to the separate elements have all possible phases, we may add to each other all these expressions. We are thus led back to the result contained in (36).

§ 14. Whatever be the nature of the processes in the interior of an element of volume, by which the radiation is caused, they can

undoubtedly be considered as determined by the state of the matter contained within the element; for this reason an electromotive force equivalent to those processes can only depend on quantities determined by that state; it cannot be altered by changing the state of the system outside the element considered, or the form and magnitude of the whole body. The formula (40), which indeed is determined by the state of things within the element **s**, must therefore be applied to an element of volume of all ponderable matter. It will be clear also that we have to add the following formulae for the amplitudes of the electromotive forces in the directions of y and z,

$$a_2 = \frac{4\pi c}{n}\sqrt{\frac{2ka_2\,dn}{\mathbf{s}}}\quad,\quad a_3 = \frac{4\pi c}{n}\sqrt{\frac{2ka_3\,dn}{\mathbf{s}}} \quad . \quad . \quad (41)$$

As to the phases of the three electromotive forces, we shall suppose them not only to change irregularly from one element to another, but also to be mutually independent in one and the same element, so that the phase-differences between the three forces have very different values in neighbouring infinitely small spaces. In virtue of this assumption the intensities of the radiation due to the different causes may be added to each other.

Till now we have only accounted for the flow of energy (36), a part of the total flow represented by (35). We shall show in the next paragraph that the remaining part

$$\frac{k\,\mathrm{S}\,a_2\,g^2\,\omega'\,dn}{c\,r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

is precisely the radiation brought about by the electromotive forces we have supposed to exist in the direction of OY, and that the forces acting in that ot OZ cannot give rise to a radiation across the element ω'. After having proved these propositions, we may be sure that, as far as the electric vibrations parallel to OX are concerned, the plate has exactly the emissivity that is required by KIRCHHOFF's law. Of course, the same will be true for the vibrations in the direction of OY.

§ 15. It may be immediately inferred from the theorem of § 7, *a* that the electromotive forces applied to the plate in the direction of OZ, i. e. perpendicularly to the surfaces, cannot contribute anything to the radiation we have considered. Indeed, we know already that an electromotive force $\mathfrak{E}_{ex}$ existing in the aether at the point P can produce no current $\mathfrak{C}_z$ in the plate; consequently, an electromotive force $\mathfrak{E}_{ez}$ in the plate cannot cause a current $\mathfrak{C}_x$ at the point P.

As to the effect produced by the electromotive force with the

amplitude a_2 acting in the direction of OY, this may be found by a reasoning similar to that we have used in §§ 12 and 13. Let us suppose for a moment an electromotive force of the same direction and intensity to exist in an element of volume $\mathbf{s}$ of the aether near the point P. The amplitude of the electric force $\mathfrak{E}_x$ in the aether immediately before the plate will then be (cfr. 38)

$$\frac{a_2 n^2 \mathbf{s}}{4\pi c^2 r},$$

and that of the current $\mathfrak{C}_y$ in the plate

$$\frac{a_2 n^2 \mathbf{s} g}{4\pi c^2 r}.$$

If follows from this that, if the element $\mathbf{s}$ in the plate is the seat of an electromotive force $\mathfrak{E}_{ey}$, with amplitude a_2, the current $\mathfrak{C}_x = \dot{\mathfrak{E}}_x$ at the point P will have this same value. The amplitude of the electric force $\mathfrak{E}_x$ will be

$$b' = \frac{a_2 n \mathbf{s} g}{4\pi c^2 r} = \frac{g}{cr}\sqrt{2k \mathbf{s} a_2 dn}$$

and the corresponding radiation across the element ω'

$$\frac{1}{2} c b'^2 \omega' = \frac{k \mathbf{s} a_2 g^2 \omega' dn}{cr^2}.$$

This leads immediately to the expression (42).

§ 16. We are now in a position to form an idea of the state of radiation in a system of bodies of any kind. After having divided them into elements of volume $\mathbf{s}$, and after having determined the principal directions at every point, we conceive in each element the electromotive forces whose amplitudes are determined by (40) and (41), the phases of all these forces being wholly independent of each other. In representing to ourselves the state of things obtained in this way, we must keep in mind:

1st. that the principal directions and the coefficients a_1, a_2, a_3 will, in general, change from point to point and will depend on the frequency n.

2ndly. that for each frequency n or rather for each interval dn of frequencies, we must assume electromotive forces of the intensity we have defined in what precedes, all these forces existing simultaneously.

We shall now show that, if the temperature is uniform throughout the system, the condition for the equilibrium of radiation will be fulfilled in virtue of our assumptions. Of course, it will suffice to prove this proposition for a single interval of frequencies dn.

Let $\mathfrak{s}$ and $\mathfrak{s}'$ be two elements of volume, arbitrarily chosen, h one of the principal directions of the first element, h' one of the principal directions of the other, a_h and $a'_{h'}$ the coefficients relating to these directions.

In virtue of the electromotive force $\mathfrak{E}_{eh}$ acting in $\mathfrak{s}$ in the direction h, there will be in $\mathfrak{s}'$ in the direction h' a current $\mathfrak{C}_{h'}$ with a certain amplitude $(\mathfrak{C}_{h'})$; by (31) the development of heat corresponding to this current will be per unit of time

$$\frac{1}{2}\, a'_{h'} (\mathfrak{C}_{h'})^2 \mathfrak{s}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Similarly, we may write

$$\frac{1}{2}\, a_h (\mathfrak{C}'_h)^2 \mathfrak{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

for the heat developed in $\mathfrak{s}$ on account of the current $\mathfrak{C}'_h$ produced in this element in the direction h by the electromotive force acting in $\mathfrak{s}'$ in the direction h'.

Since each of the three electromotive forces in $\mathfrak{s}$ calls forth a current in the element $\mathfrak{s}'$ in each of its principal directions, there will be in all nine expressions of the form (43). These must be added to each other, as may be seen by observing that the total development of heat, represented by (31), is the sum of three parts, each belonging to one of the components of the current and that the three electromotive forces in $\mathfrak{s}$ are mutually independent. The sum of the nine quantities will be the total amount of heat $\mathfrak{s}'$ receives from $\mathfrak{s}$, and in the same way we must take together nine quantities of the form (44), if we wish to determine the amount of heat transferred from $\mathfrak{s}'$ to $\mathfrak{s}$. We shall have proved the equality of the mutual radiations between the two elements, if we can show that for any two principal directions, the expressions (43) and (44) have the same value.

Let us call a_h and $a'_{h'}$ the amplitudes of the electromotive forces originating the currents whose thermal effects have been represented by (43) and (44). Then, in accordance with (40) and (41),

$$a_h = \frac{4\pi c}{n} \sqrt{\frac{2 k a_h dn}{\mathfrak{s}}}, \quad a'_{h'} = \frac{4\pi c}{n} \sqrt{\frac{2 k a'_{h'} dn}{\mathfrak{s}'}} \quad . \quad . \quad (45)$$

Now, by the general theorem of § 7, a the amplitudes $(\mathfrak{C}_{h'})$ and $(\mathfrak{C}'_h)$ in (43) and (44) are proportional to $a_h \mathfrak{s}$ and $a'_{h'} \mathfrak{s}'$. Taking into account the formula (45), we infer from this

$$(\mathfrak{C}_{h'})^2 : (\mathfrak{C}'_h)^2 = a_h^2 \mathfrak{s}^2 : a'^2_{h'} \mathfrak{s}'^2 = a_h \mathfrak{s} : a'_{h'} \mathfrak{s}',$$

an equation, which leads directly to the equality of (43) and (44).

If the system of bodies is entirely shut off from its surroundings, the equality of the mutual radiation between any two elements implies that the state is stationary.

In order to show this, we fix our attention on one particular element **s**, denoting all other elements by **s'**. By what has been said, the sum w_1 of all quantities of heat which **s** receives from the elements **s'** will be equal to the sum w_2 of the quantities of heat it gives up to them. But, if the system is isolated from other bodies, each quantity of energy lost by **s** will be found back in one of the elements **s'**; w_2 is therefore the total amount of energy radiating from **s** and the equality $w_1 = w_2$ means that **s** gains as much heat as it loses.

§ 17. We shall finally assume that the system contains a certain space which is occupied by an isotropic and homogeneous body L, perfectly transparent to the rays; we shall examine the electromagnetic state existing in this medium, if all bodies are kept at the same temperature. To this effect, we must begin by a discussion of the radiation that would take place, if the body L extended to infinity, and if it were subjected to an electromotive or magnetomotive action (§ 7) at a certain point O.

A perfectly transparent body is characterized by the absence of all thermal effects. This means that the coefficient α is zero, as appears by (30). We have therefore

$$p = -i\beta, \quad \ldots\ldots\ldots \quad (46)$$

the coefficient q being real and positive, and the equation (17) becomes

$$v = c\sqrt{\beta q n}, \quad \ldots\ldots\ldots \quad (47)$$

I shall take here the positive value.

Let us first apply to an element of volume S at the point O, which I shall take as origin of coordinates, an electromotive force $\mathfrak{E}_{ex} = ae^{int}$, but no magnetomotive force. Then

$$\mathfrak{A}_x = \frac{a\,S}{4\pi r} e^{in\left(t-\frac{r}{v}\right)}, \quad \mathfrak{A}_y = 0, \quad \mathfrak{A}_z = 0, \quad \Omega = 0.$$

What we want to know, is the amount of energy radiating from O, i. e. the flow of energy through a closed surface surrounding this point. In calculating this flow, the form and dimensions of the surface are indifferent; we shall therefore consider a sphere with O as centre and with an infinite radius r.

Then we may omit all terms in $\mathfrak{E}$ and $\mathfrak{H}$ containing the square and higher powers of $\frac{1}{r}$, and we find from (15) and (16), attending

to (46) and (47) and taking the real parts

$$\mathfrak{E}_x = \frac{a\,S\,n^2}{4\,\pi\,r\,v^2} \cdot \frac{r^2-w^2}{r^2} \cos n\left(t-\frac{r}{v}\right),$$

$$\mathfrak{E}_y = -\frac{aSn^2}{4\pi r v^2} \cdot \frac{xy}{r^2} \cos n\left(t-\frac{r}{v}\right), \quad \mathfrak{E}_z = -\frac{aSn^2}{4\pi r v^2} \cdot \frac{wz}{r^2} \cos n\left(t-\frac{r}{v}\right),$$

$$\mathfrak{H}_x = 0, \quad \mathfrak{H}_y = \frac{aSn}{4\pi r\beta c v} \frac{z}{r} \cos n\left(t-\frac{r}{v}\right), \quad \mathfrak{H}_z = -\frac{aSn}{4\pi r\beta c v} \frac{y}{r} \cos n\left(t-\frac{r}{v}\right).$$

The electric and magnetic force being known, the flow of energy through the sphere may be calculated by means of (23). Its value is

$$\frac{a^2\,S^2\,n^3}{12\,\pi\,\beta\,v^3} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (48)$$

If we perform a similar calculation in the assumption of a magnetomotive force with amplitude a, acting in the space S, the result is

$$\frac{a^2\,S^2\,n^4}{12\,\pi\,q\,v^3} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (49)$$

§ 18. Let P be a point of the body L mentioned at the beginning of the preceding paragraph, l an arbitrarily chosen direction and let us seek the amplitude $(\mathfrak{E}_l)$ of the electric current, or rather the square of the amplitude, produced by the radiating bodies, confining ourselves to the interval of frequencies dn.

We shall divide the bodies into elements of volume $\mathfrak{s}$ and we shall denote, for one of these elements lying at the point Q, by h one of the principal directions, by α_h the coefficient relating to it, and by a_h (cfr. (45)) the amplitude of the electromotive force acting in that direction.

The amplitude $(\mathfrak{E}_l)$ produced by this force at the point P is equal to the amplitude of the current $\mathfrak{E}_h$, existing in the element $\mathfrak{s}$, if an electromotive force $\mathfrak{E}_{el}$, having the amplitude $\frac{a_h\,\mathfrak{s}}{S}$ is applied to an element of volume S of the aether near P. In order to express myself more briefly, I shall understand by A the radiation that would be excited by an electromotive action at the point P in the direction l of such intensity that the product $(\mathfrak{E}_{el})$ S has the value 1. The amplitude $(\mathfrak{E}_l)$ in P, of which we have just spoken, will be found if we multiply by $a_h\,\mathfrak{s}$ the value which, in that state, $(\mathfrak{E}_h)$ would have in the element $\mathfrak{s}$. Hence

$$(\mathfrak{E}_{lP})^2 = a_h^2\,\mathfrak{s}^2\,(\overset{\Lambda}{\mathfrak{E}}_{hQ})^2 = \frac{32\,\pi^2\,c^3\,k\,\alpha_h\,\mathfrak{s}\,dn\,(\overset{\Lambda}{\mathfrak{E}}_{hQ})^2}{n^3}$$

$$= \frac{64\,\pi^2\,c^2\,k\,dn}{n^2}\,\overset{\Lambda}{w}_h, \quad (50)$$

if we write w_h^A for the development of heat in the element **s**, which, in the state A, is due to the current in the principal direction h.

Now, starting from the expression (50), we shall obtain the total value of $(\mathfrak{E}_{lP})^2$ by an addition, in which all elements **s**, each with its three principal directions, must be taken into account. In a system, completely shut off from surrounding bodies, Σw_h^A will be the total amount of energy, emitted by P in the state A; we can therefore determine it by the formula (48), putting $a\,S = 1$. This leads to the result

$$(\mathfrak{E}_{lP})^2 = \frac{16\,\pi\,k\,c^3\,n\,dn}{3\,\beta\,v^3}.$$

In the same way, using the theorem of § 7, b and the expression (49), I find

$$(\mathfrak{B}_{lP})^2 = \frac{16\,\pi\,k\,c^2\,n^2\,dn}{3\,q\,v^3}.$$

These results being independent of the place of the point P and the choice of the direction l, we come to the conclusion that the state of things is the same in all parts of the medium L and that both the electric and the magnetic vibrations take place with equal intensities in all directions. The amount of the electric and magnetic energy per unit of volume is now easily found. According to § 4 the first is

$$\frac{1}{4}\,n\,\beta\,[(\mathfrak{D}_x)^2 + (\mathfrak{D}_y)^2 + (\mathfrak{D}_z)^2],$$

for the value of which one finds

$$\frac{4\,\pi\,k\,c^3\,dn}{v^3},$$

by remembering that for every direction l,

$$(\mathfrak{D}_l)^2 = \frac{1}{n^2}\,(\mathfrak{E}_l)^2.$$

The magnetic energy may likewise be determined. Referred to unit volume it has the value

$$\frac{1}{4}\,q\,[(\mathfrak{B}_x)^2 + (\mathfrak{B}_y)^2 + (\mathfrak{B}_z)^2],$$

and this is easily calculated, since for every direction l,

$$(\mathfrak{B}_l)^2 = \frac{1}{n^2}\,(\mathfrak{B}_l)^2.$$

The result is that the two kinds of energy are distributed over the body l with equal densities. This has been known for a long

time, as has also been the rule implied in our formulae, that these densities are inversely proportional to the cube of the velocity of propagation v. It must further be noticed that, if the medium L is aether, the density of the energy of the radiation becomes

$$\frac{8\pi k\, dn}{c}.$$

This agrees with the meaning we have originally attached to the coefficient k (§ 11).

§ 19. There is one point in the foregoing considerations that may at first sight seem strange, viz. that the intensity of the electromotive forces we have imagined should depend on the magnitude of the elements of volume **s**. It must be kept in mind however, that these forces have no real existence, and that we do not pretend to have found something concerning the causes by which the phenomena are produced. That the magnitude of the electromotive forces must be taken inversely proportional to the square root of the volume of **s** is simply a consequence of our assumption that the force has the same phase in all points of such an element. For a given amplitude of the electromotive force, the radiation would therefore be proportional to $\mathbf{s}^2$, and we had to make such assumptions concerning that amplitude, that the radiation became proportional to **s** itself.

In connection with these remarks it must be observed that we have no reasons for ascribing to the dimensions of the elements of volume some particular value. These dimensions are indifferent as long as we consider only the radiation at finite distances and the transfer of energy between neighbouring molecules lies outside the theory I have here developed.

Physiology. — *"On the ability of distinguishing intensities of tones"*. By Prof. H. ZWAARDEMAKER. (Report of a research made by A. DEENIK.)

The "Unterschiedsschwelle" for impulsive sounds (dropping bullets and hammers) has been studied frequently and many-sidedly, but regarding the "Unterschiedsschwelle" for intensities of tone we have had at our disposal till now only some information communicated by M. WIEN in his thesis.

M. WIEN found the value of the "Unterschiedsschwelle" for the three tones, to which he limited his investigation to be as follows: for a average 22.5% (with 18.2 and 27 for extremes) for e' 17.6%

(one determination) for a' average 14.4% (with 10.8 and 22.5 for extremes). It appeared desirable to perform such an investigation through the whole scale and to establish it in other regards also on larger foundations. At my request Mr. A. DEENIK has executed a very great number of observations of this kind, and I take the liberty to communicate his results here in short, and refer the reader to an ample description in a thesis on this subject by Mr. DEENIK which will soon be published.

Experiments with the tuning-fork.

A tuning-fork kept vibrating by electro-magnetism is started in a room at the side of the sound-free cabinet of the physiological laboratory and is kept vibrating at a fixed amplitude. This amplitude may be measured microscopically by means of the triangle of GRADENIGO. Normal to the axis of this tuning-fork a circle divided into grades is placed, to which two hearing-tubes are attached in such a way, that their radial prolongations cut the axis of the tuning-fork in the tuning-centre. These hearing-tubes can be moved along the whole circumference of the scale, and can be brought at pleasure into the interference-planes of KIESSLING, in the planes of maximum-sound or between.

The hearing-tubes are led into the interior of the sound-free cabinet by means of thick-walled caoutchouc tubes which were still further acoustically isolated. There by means of a T—tap alternately the one or the other of the tubes may be listened at or perfect acoustic rest can be obtained by bringing the tap into a closed position.

An assistant now displaces one of the hearing-tubes, while the other hearing-tube is fixed in the plane of maximum sound, every time through some grades at a time into the direction to the interference plane of KIESSLING till a distinct difference has been signalised by the investigator (descending method). After the position of the tube has been read this is pushed on and then brought back in the same way till the investigator observes that the existing difference in intensity becomes indistinct (ascending method). Again the position of the tube is read off and the average is taken.

The observations take place in the above mentioned way "unwissentlich" and at five succeeding times. From the ten figures obtained in this way the average is taken at last, which indicates in grades of the scale a lowest "Unterschiedsschwelle" for the concerned amplitude.

To be able to transpose these angle-values into absolute values,

in the sound free cabinet, which has been internally covered with trichopiese, the greatest distance at which sound is still perceptible is determined for the intensity of sound in the maximumplane and for that in the discovered "Unterschiedsschwelle" plane. If we accept that in case of absence of reverberations, as we may suppose here, the sound intensities decrease proportionate to the quadrates of the distances, the sound intensities stand mutually in the same proportion as the quadrates of those distances. If we call the distance at which the tone sound is perceptible in the plane of maximum sound r and that for the somewhat weaker sound r_u, then the quotient $\frac{r^2 - r_u^2}{r_u^2}$ represents evidently the "Unterschiedsschwelle", which in this case may be indicated as "untere Unterschiedsschwelle" because the stimulation distinguished from the chief is taken weaker than the chief stimulation.

TABLE I. Experiments with the tuning-fork.

Tone level.	Amplitude in microns	$\frac{\triangle r}{r}$	'Unterschiedsschwelle" (average).
c^1	640	0.29589	
	800	0.34429	33.2 %
	1040	0.35657	
c^2	20	0.22698	
	40	0.26932	
	70	0.29825	
	100	0.30835	29.3 %
	150	0.31003	
	200	0.31540	
	300	0.32006	
c^3	2	0.23435	
	2	0.20243	19.5 %
	2	0.14865	

Experiments with organ-pipes.

An accurately tuned, wide, covered, wooden organ-pipe is placed in a felt tent in a room at the side of the soundfree cabinet in such

a way that the sound may be listened to through a caoutchouc tube in the cabinet. This organ-pipe is permanently blown by air which was supplied by a presspump driven by water and afterwards dried with chloride of calcium. The supply of this air takes place along a long system of leaden tubes, which shows inside the cabinet a division into two parts and afterwards a reunion. To this two separate branches by micrometer screws removable diaphragm openings of AUBERT are attached, which may be widened or narrowed at pleasure. The reunion takes place in a T-tap, which may also be directed by the investigator, and down the current are placed the necessary measuring apparatus for determining the pressure and volume of the air passing to the organ-pipe. These measuring apparatus are placed within the reach of the investigator, so that he himself can do the reading off.

The investigator arranges in the first place the width of the two diaphragm-openings in such a way that the sound may be called equal in the two positions of the tap. Then he enlarges one of the diaphragmata (the other remains constant) till a distinct difference is perceived (ascending method). This he does five times. After this the difference between the two tone intensities, which were alternately listened to, was enlarged and the diaphragm position was ascertained by descending at which the difference became indistinct (descending method). This again was done five times. The same takes place conformally in narrowing the diaphragm-openings. So the first series leads to a "obere" the second to an "untere Unterschiedsschwelle". The determinations which were made for each tone with two chief intensities have evidently taken place "wissentlich" in this way. At last a pressure and volume determination of the supplied air is made for the found diaphragm widths. The first takes place by means of a watermanometer, which for increasing sensibility has been put sloping; the second with an aerodromometer[1]). The energy offered to the organ-pipe could be calculated with the usual formula $e =$ air-volume $\times$ pressure $\times$ 981. This number, multiplied by a constant factor, different for each pipe, indicates the acoustic energy.

As in the expression of the "prozentische Unterschiedsschwelle" $\frac{\Delta R}{R}$ the constant factor occurs both in the numerator and the denominator, the constant factor of the organ-pipe falls away from the further calculation and we can also come to a trustworthy result of the "prozentische Unterschiedsschwelle" without its preceding

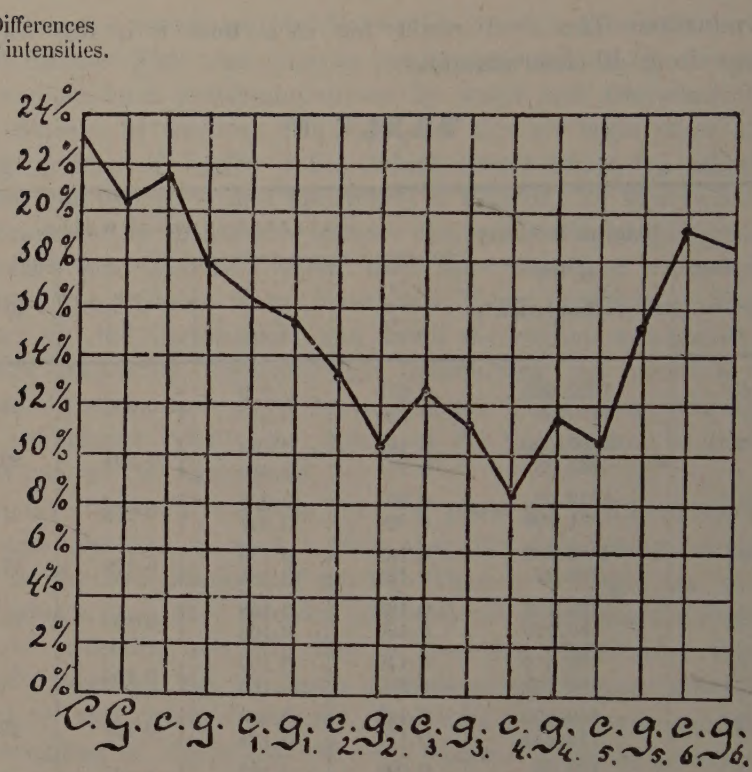
[1]) Arch. f. (Anat. u.) Physiologie 1902 supplement. p. 417.

determination. The final result for each tone is in this way the average from 40 determinations.

TABLE II.

Tone level	Relative intensity of the chief stimulation [1])	Unterschiedsschwelle. $\frac{\Delta^o r}{r}$	$\frac{\Delta^u r}{r^u}$	$\frac{\Delta r}{r}$	in %
C	1392 408	0.237	0.219	0.232	23.2
	1120.55	0 236	0.237		
G	1560.168	0.199	0.199	0.204	20.4
	1243.35	0.211	0.210		
c	1213.63	0.201	0.221	0.218	21.8
	861.798	0.224	0.227		
g	1412.200	0.183	0.179	0.179	17.9
	788.97	0.173	0.184		
c_1	107.112	0.162	0.158	0.163	16.3
	86.129	0.166	0.168		
g_1	132.411	0.142	0.152	0.154	15.4
	104 725	0.157	0.166		
c_2	140.800	0.143	0.108	0.131	13.1
	114.444	0.137	0.134		
g_2	139.96	0.098	0.112	0.105	10.5
	101.152	0.108	0.105		
c_3	135.976	0.104	0.132	0.125	12.5
	101.764	0.126	0.138		
g_3	134.552	0.114	0.108	0 112	11.2
	98.424	0.120	0.108		
c_4	251 01	0.077	0.082	0.085	8.5
	139.438	0.081	0.101		
g_4	332.072	0.117	0.122	0.117	11.7
	230.888	0.107	0.121		
c_5	424.636	0.108	0.114	0.107	10.7
	280.908	0.100	0.107		
g_5	295 68	0.160	0.155	0.154	15.4
	218.621	0 157	0.145		
c_6	260.100	0 188	0.200	0.192	19.2
	183.272	0.204	0.178		
g_6	580.190	0.164	0.191	0.188	18.8
	480.720	0.171	0.229		

[1]) For the calculation of the absolute intensity the number of the second column must still be multiplied by a constant factor which however falls away in the calculation of the "Unterschiedsschwelle" and is of no consequence.



Smallest perceptible difference of intensity by the scale.

CONCLUSION.

1. From the results of the experiments with the tuning-fork proceeds that the law of WEBER is valuable, when taken in a general way, but not exactly for the investigated middle-strong and weak intensities.

2. From the results of the organ-pipes proceeds that the most favourable "Unterschiedsschwelle" is found with c^4 and that from there to the ends the power of distinguishing differences in intensities decreases rather regularly.

ERRATUM.

p. 380 line 7 for 0,990 1,03 read 1,02 1,04.

(November 22, 1905).